Perfectly Matched Absorbing Boundary Conditions Based on Anisotropic Lossy Mapping of Space

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Abstract—**An absorbing boundary condition (ABC), based on the PML ABC of Berenger, used in the frequency domain to terminate the computational grid in electromagnetic scattering simulations is presented. Making use of an impedance-matched lossy layer, with directionally-dependent electric and magnetic conductivity, this ABC is independent of frequency and almost independent of incident angle. Thus it can be placed very close to a scatterer, minimizing the usual buffer of required, uninteresting computational space, and so reduce computer storage and CPU time. With this novel formulation, the ABC can be specified in three dimensions in almost the same manner as the standard FDFD equations, with only a small percentage of increased software overhead.**

I. Introduction and Background

THE GOAL of all local absorbing boundary condition computational lattice terminations is to simulate scattered waves propagating outward and prevent reflection artifacts. ABC's must absorb waves with wavefronts incident at all angles, and for practical use in time domain computations, they must prevent reflections for all frequencies. Also, it is desirable for these boundaries to be computationally simple, using repeated simple calculations to take advantage of nearest neighbor communications in massively parallel computer platforms.

Several methods based on pseudo-annihilation algorithms have been proposed and are currently in use. These are extensively reviewed by Givoli [1]. Although it is possible to increase the order of these ABC's to improve their absorption from wider angles, they become more complex, requiring considerably more storage and calculation. Often, the higher-order ABC's involve repeatedly inverting large matrices, particularly ill-suited for parallel processing.

An alternative has been suggested [2], [3], which uses a lossy layer to absorb scattered field. By choosing a hypothetical medium with electric and magnetic loss in inverse ratio to that of the square of the impedance of free space ($\sigma/\sigma_m = \epsilon_0/\mu_0$), waves that are transmitted into this layer maintain the same propagation characteristics, with the same phase velocity and orientation, but with frequency-independent attenuation. Given sufficiently thick material, the wave decays to a negligible level. Efficient transmission into this medium from wide angles is accomplished by roughening its boundary into an anechoic chamber-like sawtooth configuration.

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A very recent paper by Berenger [4], with validations and extensions by others [5], suggests an alternative means of ensuring transmission in a lossy layer from wide incident angles. Instead of roughening the layer surface, this Perfectly Matched Layer (PML) method uses a lossy layer which retains only the electric and magnetic conductivities (with the same ratio as above) in the components of Maxwell's Equations which involve derivatives taken with respect to the direction normal to the boundary. The governing curl equation for the transverse field component is decomposed into two equations, with this field also divided into two parts. For either TE or TM calculations, four equations are needed instead of the usual three; and four field quantities must be stored and computed.

The PML method is herein generalized for any lattice termination geometry,and computations of the frequency domain version of the PML are computed. Improvements to the PML method using a tuned lossy layer termination are also presented. It should be emphasized that this is not an attempt to compare a new method to the PML, but rather to describe one possible heuristic basis for matched lossy layer ABC's.

II. Spatial Mapping for Matched Lossy Layer

A more general approach to this directionally-dependent conductive layer is to model the half-space into which the wave transmits and decays as having a lossy coordinate. If x is the coordinate normal to the layer surface, replace it with the complex spatial variable:

$$x' = x\left(1 - \mathrm{j}\,\frac{\sigma}{\omega\epsilon_0}\right) \tag{1}$$

in Maxwell's equations. For time-harmonic plane waves with angular frequency ω, this has the effect of altering the wave number/distance product in the normal direction to $k_x x' = k_x(1 - \mathrm{j}\sigma/\omega\epsilon_0)x$. A plane wave propagating in free space with direction $\overline{k}$ thus transmits into the layer at $x = 0$ by the relation:

$$e^{-j\overline{k}\cdot\overline{r}} \rightarrow e^{-j\overline{k}\cdot\overline{r}}e^{-\alpha x} \tag{2}$$

with $\alpha = \sigma\eta_0 \cos\theta$, for incidence angle θ. Notice that since both the amplitude and phase of waves are continuous across the boundary, there will be no reflection from this boundary for any incident angle. In the time domain, the corresponding equation for a plane wave propagating in the $\overline{d}$-direction can be written: $f(|\overline{d}| - ct) \rightarrow f(|\overline{d}| - ct)e^{-\alpha x}$ for any arbitrary function f. Also, since the loss (as well as the transmission coefficient) is independent of frequency, this layer is perfectly suited to time-domain calculations.

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The time-harmonic Maxwell's curl equations for TM waves (relative to the ABC boundary) in this lossy layer for $x \geq x_{\mathrm{MAX}}$ using complex remapped space of (1) are:

$$\begin{aligned} \frac{\partial H_z}{\partial y} &= \mathrm{j}\omega\epsilon_0 E_x \\ -\frac{\partial H_z}{\partial x}\left(\frac{1}{1-\mathrm{j}\sigma/\omega\epsilon_0}\right) &= \mathrm{j}\omega\epsilon_0 E_y \\ \frac{\partial E_y}{\partial x}\left(\frac{1}{1-\mathrm{j}\sigma/\omega\epsilon_0}\right) - \frac{\partial E_x}{\partial y} &= -\mathrm{j}\omega\mu_0 H_z \end{aligned} \quad (3)$$

and similarly for $-H_y$ replacing H_z, with y and z coordinates exchanged.

The TE equations are similar, with $(-\overline{E}, \overline{H}, \mu_0, \epsilon)$ replacing $(\overline{H}, \overline{E}, \epsilon_0, \mu_0)$. Ordinarily, the loss term in Faraday's law in the third equation of (3) would involve a magnetic conductivity σ_m. But for the impedance match condition, $\sigma_m/\mu_0 = \sigma/\epsilon_0 = 1/\tau$ (τ being the dielectric relation time constant), so all equations are written in terms of σ.

With suitable manipulation, Berenger's directionally-dependent conductivity equations [4] can be cast in the frequency domain, giving the three equations of (3). However, the complex spatial mapping approach is more general, conceptually more intuitive, and it avoids the artificial splitting of the H_z term into two parts.

The FDFD equations follow as usual, with forward differencing discretization $(x, y, z, t) = (i\Delta, j\Delta, k\Delta, n\Delta t)$, and $\mathrm{j} = \sqrt{-1}$. For example, the third equation of (3) becomes:

$$\begin{aligned} H_z(i,j,k) = \Bigg\{ & E_y(i-\tfrac{1}{2}, j, k) - E_y(i+\tfrac{1}{2}, j, k) \\ & + [E_x(i, j+\tfrac{1}{2}, k) - E_x(i, j-\tfrac{1}{2}, k)] \\ & \cdot \left(1 - \mathrm{j}\frac{\sigma}{\omega\epsilon_0}\right)\Bigg\} \Big/ \\ & [(\mathrm{j}\omega\mu_0 + \sigma\eta_0^2)\Delta] \end{aligned} \quad (4)$$

A similar set of equations applies for the corresponding H_y equations, the dual TE equations, and for other half-space boundaries, by merely replacing the x-coordinate with the appropriate normal coordinate. For non-Cartesian geometries, the same principle applies. For example, a circular ABC lossy layer would make use of polar coordinates with the loss term $(1 - \mathrm{j}\sigma/\omega\epsilon_0)$ associated with the radial coordinate, ρ. This modification changes the usual free-space FDFD equation coefficients by only adding terms involving σ. These coefficients are computed before any of the iterations are performed, so there is almost no increase in computer overhead for calculating the propagation in this lossy layer.

The simplicity of the frequency domain formulation of the directionally dependent lossy layer unfortunately does not translate to the time domain quite as well. Multiplying (3c) by $\mathrm{j}\omega + \sigma/\epsilon_0$, and transforming yields the corresponding time-domain equation: $[(\partial^2 E_x/\partial y\,\partial t) + (\sigma/\epsilon_0)(\partial E_x/\partial y)] - (\partial^2 E_y/\partial x\,\partial t) = \mu_0(\partial^2 H_z/\partial t^2) + \sigma\eta_0^2(\partial H_z/\partial t)$. Because of the second-order derivatives, this time-domain version of the modified Faraday's law is awkward to use for FDTD: several previous time values of both electric and magnetic fields must be stored. Thus, Berenger's original formulation with artificially split H_z would be computationally more useful.

III. Lossy Layer Termination

The lossy layer itself can be optimally terminated. Berenger [4] terminates the PML lossy layer with a simple perfectly conducting wall, which introduces 100% reflection (the worst possible boundary) at the lattice termination. For normally incident waves ($\theta = 0$), the reflection coefficient R is small enough to make this reflection insignificant. For glancing incidence, however, the reflection coefficient is reduced to $R^{\cos\theta}$, and the large termination reflection dominates the absorption of the entire layer. Standard ABC techniques can be used at this termination to help reduce the large reflections for glancing incidence.

In the frequency domain, the transverse electric and magnetic fields at the end of the lossy layer can be constrained to have a wave impedance ratio of the plane wave with any given propagation direction. This termination would be a perfect match for the wave incident on the ABC with this particular angle θ. It will slightly reflect all other incident waves.

It is noted that in the time domain, a similar impedance termination can be used—or a lossy version of the Engquist Majda ABC time [6], with propagation direction and decay rate adjusted to match the desired angle and layer loss—may be implemented.

IV. FDFD Simulation in a Parallel Plate Waveguide

To test the frequency domain version of this PML ABC, a parallel plate waveguide excited by single TM modes is numerically stimulated. The waveguide is specified with guide wavelength 10Δ and is terminated with an 8 grid-point lossy layer, with physical length d_x, and a parabolic conductivity profile which is zero at the free-space interface, and has a maximum value of $\sigma_m = 8/\eta d_x$ at its other edge (corresponding to PML(8, P, 0.5) in Berenger's notation [4], or 0.5% reflection). It was found that increasing the maximum conductivity generated excessive reflections from the first few lossy layer points, so PML(8, P, 0.0 001) could not be successfully implemented in the frequency domain. Using mesh-refinement or multigrid techniques would improve the FDFD results, and will be considered in a subsequent report. The layer is terminated with the impedance boundary condition discussed above, tuned to perfectly match waves at either $\theta_m = 45°$ or $75°$ incidence, corresponding to the mode with characteristic impedance: $Z = \beta_x/\omega\epsilon_0 = \eta_0 \cos\theta_m$. For a staggered FDFD grid, $E_y(7.5\Delta)/H_z(8\Delta) = Ze^{\sigma_m\Delta/2}$.

Simulation results are given in Table I. Three cases are shown: the new formulation, optimized for 45° and 75°, and the Fourier Transformed Berenger equations. For each angle, the VSWR is lower in each of the new cases than it is for the reference case. The corresponding reflection coefficient magnitudes for the 75° case for 0°, 45°, and 75° degrees, are: 0.005, 0.004, and 0.0003, respectively. Note that even though these results are for conductivity choice PML(8, P, 0.5) the 75° value is two orders of magnitude lower than the *best*

TABLE I
VSWR FOR A PARALLEL PLATE WAVEGUIDE WITH AN 8 GRID-POINT LAYER WITH PARABOLIC TAPERED CONDUCTIVITY, CORRESPONDING TO PML(8, P, 0.5) [4] FOR BERENGER'S FORMULATION

Angle	VSWR		
	No Termination (Berenger)	Termination Optimized at 45°	Termination Optimized at 75°
0°	1.016	1 014	1.014
45°	1 027	1 010	1 016
60°	1 090	1 019	1 027
75°	1 412	1 172	1 003

corresponding time domain value, 0.03, reported for PML(8, P, 0.0001).

The impedance matched layer termination clearly reduces reflections from the layer. However, even at the optimization angle, the reflection is not reduced to zero. Reflections from intermediate lossy layer grid point discontinuities dominate the reflection from the entire layer, as, for example, in the 45° case where the VSWR at 45° is greater than the minimum for the 75° case.

The performance of the lossy layer ABC *decreases* for higher resolution field sampling. That is, for a fixed number of grid points in the layer, the greater the number points per wavelength, the worse the reflection coefficient. This results from the increased decay rate per wavelength in the lossy layer. The layer characteristics are independent of frequency, but the apparent severe decay overwhelms the FDFD algorithm. Mathematically, this occurs because the conductivity discontinuity factor $\sigma/\omega\epsilon_0 = \sigma\eta/k$ becomes large compared to unity for low frequencies, when $k\Delta$ is small. If instead the layer thickness were specified as a fixed percentage of the wavelength, decreasing the sample size Δ could yield arbitrarily low reflections. Although it is inappropriate to compare time and frequency domain performances, the best FDFD result for normal incidence is worse than the best FDTD results, because of both the gradual rise times for waves crossing conductivity discontinuities, and the exponential finite time differencing [4] allowed in FDTD. Exponential differencing is not possible with FDFD since there is no angle-independent spatial decay rate parameter corresponding to the time rate τ.

V. CONCLUSION

A generalization of the Berenger PML [4] ABC has been formulated in terms of mapping the surface normal spatial coordinate into a complex (lossy) form. This representation applies to two- and three-dimensional geometries, and avoids using the artificial separation of the transverse field into two parts, which in the frequency domain avoids the required additional equations, computation, and data storage.

The FDFD form of this PML ABC has been tested for a parallel-plate waveguide, and has given excellent results. Using a matched impedance boundary condition to terminate the lossy layer in either the frequency or the time domain for a single extreme angle of incidence compensates for the reduced attenuation of the layer for glancing incidence. This concept can be applied to all lossy layer ABC's, in both the FDTF and FDTD without increasing computational overhead.

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